

A NOTE ON SESHADRI CONSTANTS OF LINE BUNDLES ON HYPERELLIPTIC SURFACES

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ABSTRACT. We study Seshadri constants of ample line bundles on hyperelliptic surfaces. We obtain new lower bounds and compute the exact values of Seshadri constants in some cases. Our approach uses results of F. Serrano (1990), B. Harboure and J. Roé (2008), F. Bastianelli (2009), A.L. Knutsen, W. Syzdek and T. Szemberg (2009).

1. INTRODUCTION

Seshadri constants measure how positive a line bundle is. They were introduced in 1992 by J.P. Demailly in [De1992] as an attempt to tackle the famous Fujita conjecture. The conjecture has not been proven but Seshadri constants soon became an object of study on their own.

Giving exact values or just estimating Seshadri constants is very hard, even in case of line bundles on algebraic surfaces, see e.g. [Ba1999]. There exists an upper bound for Seshadri constant of a line bundle at points x_1, \dots, x_r on a smooth projective n -dimensional variety X , namely $\varepsilon(L, x_1, \dots, x_r) \leq \sqrt[n]{\frac{L^n}{r}}$. Therefore it is interesting to look for lower bounds.

There are several results concerning Seshadri constants on surfaces with Kodaira dimension zero. Let us recall some of them. In appendix to [Ba1998], Th. Bauer and T. Szemberg give upper bound for the global Seshadri constant of an ample line bundle on an abelian surface and as a corollary obtain that the Seshadri constant of such a line bundle is always rational. In [Ba1997] Th. Bauer computes Seshadri constants on all $K3$ surfaces of degree 4. This result is extended by C. Galati and A.L. Knutsen in [GaK2013] who compute Seshadri constants on $K3$ surfaces of degrees 6 and 8. Earlier in [K2008] A.L. Knutsen estimates Seshadri constants on $K3$ surfaces with Picard number 1. T. Szemberg in [Sz2001] proves that the global Seshadri constants on Enriques surfaces are always rational and also provides the lower bound for Seshadri constant at an arbitrary point. Up to our knowledge, Seshadri constants have not been studied on hyperelliptic surfaces before.

We estimate Seshadri constants on hyperelliptic surfaces, in some cases we compute their exact values. The paper is organised in the following way — in Theorem 3.1 we compute the global Seshadri constant of a line bundle of type (1,1) on a hyperelliptic surface of an arbitrary type, next in Proposition 3.3 we point out a hyperelliptic surface type and a point at which a Seshadri constant of a line bundle of type (1,1) is strictly greater than 1. In Theorem 3.4 we compute the global Seshadri constant of a arbitrary ample line bundle on hyperelliptic surface of type 1, and in Theorem 3.5 we provide a lower bound for this constant on hyperelliptic surfaces of types 2-7. Finally, in Theorem 3.6 we estimate from below the

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multi-point Seshadri constant of an ample line at r very general points on hyperelliptic surfaces.

2. NOTATION AND AUXILIARY RESULTS

Let us set up the notation and basic definitions. We work over the field of complex numbers \mathbb{C} . We consider only smooth reduced and irreducible projective varieties. By $D_1 \equiv D_2$ we denote the numerical equivalence of divisors D_1 and D_2 . By a curve we understand an irreducible subvariety of dimension 1. In the notation we follow [Laz2004].

Let X be a smooth projective variety and L a nef line bundle on X . We recall the definition of a Seshadri constant.

Definition 2.1. (1) The Seshadri constant of L at a given point $x \in X$ is the real number

$$\varepsilon(L, x) = \inf \left\{ \frac{LC}{\text{mult}_x C} : C \ni x \right\},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through x .

(2) The global Seshadri constant of L is defined to be

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x).$$

Let x_1, \dots, x_r be pairwise distinct points. The notion of a Seshadri constant of a line bundle at a point may be generalised to r points in the following way:

Definition 2.2. The multi-point Seshadri constant of L at x_1, \dots, x_r is the real number

$$\varepsilon(L, x_1, \dots, x_r) = \inf \left\{ \frac{LC}{\sum_{i=1}^r \text{mult}_{x_i} C} : \{x_1, \dots, x_r\} \cap C \neq \emptyset \right\},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through at least one of the points x_1, \dots, x_r .

For a fixed line bundle L the function $(x_1, \dots, x_r) \mapsto \varepsilon(L, x_1, \dots, x_r)$ is constant for points in very general position, moreover its value for points not lying in very general position does not exceed the value for points in very general position — see [Laz2004], Example 5.1.11. We denote the Seshadri constant of L at r points in very general position by $\varepsilon(L, r)$.

Let $\alpha_0(L, m_1, \dots, m_r)$ denote the least degree LC of an irreducible curve C passing through r points in general position with multiplicities m_1, \dots, m_r . Let $m^{[l]} = \underbrace{(m, \dots, m)}_{l \text{ times}}$. Then the

following theorem holds:

Theorem 2.3 (Harbourne, Ro  , [HR2008], Theorem 1.2.1). *Let L be a big and nef line bundle on a smooth projective surface. Let $r \in \mathbb{N}$, $r \geq 2$, let $\mu \in \mathbb{R}$, $\mu \geq 1$. If*

(1) for every $m \in \mathbb{N}$ such that $1 \leq m < \mu$ we have

$$\alpha_0(L, m^{[r]}) \geq m \sqrt{L^2 \left(r - \frac{1}{\mu} \right)}$$

and

(2) for every $m \in \mathbb{N}$ such that $1 \leq m < \frac{\mu}{r-1}$ and if for every $k \in \mathbb{Z}$ such that $k^2 < \frac{r}{r-1} \min\{m, m+k\}$ we have

$$\alpha_0(L, m^{[r-1]}, m+k) \geq \frac{mr+k}{r} \sqrt{L^2 \left(r - \frac{1}{\mu}\right)},$$

then

$$\varepsilon(L, r) \geq \sqrt{\frac{L^2}{r}} \sqrt{1 - \frac{1}{r\mu}}.$$

For more background on Seshadri constants we refer to [PSC2009].

Now let us recall the definition of a hyperelliptic surface.

Definition 2.4. A hyperelliptic surface S (sometimes called bielliptic) is a surface with Kodaira dimension equal to 0 and irregularity $q(S) = 1$.

Alternatively ([Bea1996], Definition VI.19), a surface S is hyperelliptic if $S \cong (A \times B)/G$, where A and B are elliptic curves, and G is an abelian group acting on A by translation and acting on B , such that A/G is an elliptic curve and $B/G \cong \mathbb{P}^1$; G acts on $A \times B$ coordinatewise. Hence we have the following situation:

$$\begin{array}{ccc} S \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \Psi \downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

where Φ and Ψ are natural projections.

Hyperelliptic surfaces were classified at the beginning of 20th century by G. Bagnera and M. de Franchis in [BF1907], and independently by F. Enriques i F. Severi in [ES1909-10]. They showed that there are seven non-isomorphic types of hyperelliptic surfaces. Those types are characterised by the action of G on $B \cong \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$ (for details see eg. [Bea1996], VI.20). For every hyperelliptic surface we have that the canonical divisor K_S is numerically trivial.

In 1990 F. Serrano in [Se1990] characterised the group $\text{Num}(S)$ for each of the surface's type:

Theorem 2.5 (Serrano). *A basis of the group of classes of numerically equivalent divisors $\text{Num}(S)$ for each of the surface's type and the multiplicities of the singular fibres in each case are the following:*

Type of a hyperelliptic surface	G	m_1, \dots, m_s	Basis of $\text{Num}(S)$
1	\mathbb{Z}_2	2, 2, 2, 2	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2, 2, 2, 2	$A/2, B/2$
3	\mathbb{Z}_4	2, 4, 4	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	2, 4, 4	$A/4, B/2$
5	\mathbb{Z}_3	3, 3, 3	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	3, 3, 3	$A/3, B/3$
7	\mathbb{Z}_6	2, 3, 6	$A/6, B$

Let $\mu = \text{lcm}\{m_1, \dots, m_s\}$ and let $\gamma = |G|$. Given a hyperelliptic surface, its basis of $\text{Num}(S)$ consists of divisors A/μ and $(\mu/\gamma)B$. We say that L is a line bundle of type (a, b) on a hyperelliptic surface if $L \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B$. In $\text{Num}(S)$ we have that $A^2 = 0$, $B^2 = 0$, $AB = \gamma$. Note that a divisor $b \cdot (\mu/\gamma)B \equiv (0, b)$, $b \in \mathbb{Z}$, is effective if and only if $b \cdot (\mu/\gamma) \in \mathbb{N}$ (see [Ap1998], Proposition 5.2).

The following proposition holds:

Proposition 2.6 (see [Se1990], Lemma 1.3). *Let D be a divisor of type (a, b) on a hyperelliptic surface S . Then*

- (1) $\chi(D) = ab$;
- (2) D is ample if and only if $a > 0$ and $b > 0$;
- (3) If D is ample then $h^0(D) = \chi(D) = ab$.

Now we recall a bound for the self-intersection of a curve. Adjunction formula, applied to the normalisation of a curve C , implies the following formula:

Remark 2.7 (Genus formula, [GH1978], Lemma, p. 505). *Let C be a curve on a surface S , passing through x_1, \dots, x_r with multiplicities respectively m_1, \dots, m_r . Let $g(C)$ denote the genus of the normalisation of C . Then*

$$g(C) \leq \frac{C^2 + C \cdot K_S}{2} + 1 - \sum_{i=1}^r \frac{m_i(m_i - 1)}{2}.$$

Note that a curve C on a hyperelliptic surface has genus at least 1. Otherwise the normalisation of C , of genus zero, would be a covering (via Φ) of an elliptic curve A/G . This contradicts the Riemann-Hurwitz formula.

For families of curves we have Xu-type lemma. The original version of this lemma was proved by G. Xu in [Xu1995]. We will use the generalisation of the Xu Lemma obtained by A.L. Knutsen, W. Syzdek, T. Szemberg in [KSSz2009], and independently by F. Bastianelli in [Bas2009]. Let $\text{gon}(C)$ denote the gonality of a smooth curve C , i.e. the minimal degree of a covering $C \rightarrow \mathbb{P}^1$.

Lemma 2.8 (Bastianelli, [Bas2009], Lemma 2.2; Knutsen-Syzdek-Szemberg, [KSSz2009], Theorem A). *Let S be a smooth projective surface. Let U be a smooth variety. Consider a nontrivial family $\{(C_u, x_u)\}_{u \in U}$ where x_u is a very general point of S and C_u is a curve satisfying the condition $\text{mult}_{x_u} C_u \geq m$ for every $u \in U$ and for some integer $m \geq 2$. Then for a general curve C of this family*

$$C^2 \geq m(m - 1) + \text{gon}(\tilde{C}).$$

Applying the Xu-type lemma to a family \mathcal{C} of curves passing through x_1, \dots, x_r with multiplicities respectively m_1, \dots, m_r , where $m_1 \geq 2$, on a blow-up at x_2, \dots, x_r , we have the following multi-point version of the Xu-type lemma

Lemma 2.9. *For a general curve C of the family \mathcal{C} as above we have that*

$$C^2 \geq \left(\sum_{i=1}^r m_i^2 \right) - m_1 + \text{gon}(\tilde{C}).$$

Every hyperelliptic surface S is nonrational, hence for every curve $C \subset S$ we have $\text{gon}(\tilde{C}) \geq 2$ (see [KSSz2009], remarks following Theorem A).

3. MAIN RESULTS

3.1. Seshadri constants of ample line bundles on hyperelliptic surfaces. We start with computing the global Seshadri constant in the simplest case of an ample line bundle on a hyperelliptic surface, i.e. for a line bundle of type $(1, 1)$.

Theorem 3.1. *Let S be a hyperelliptic surface. Let L be a line bundle of type $(1, 1)$ on S . Then*

$$\varepsilon(L) = 1.$$

Proof. Let $C \equiv (\alpha, \beta)$ denote an irreducible curve passing through a given point $x \in S$ with multiplicity m , $m \geq 1$. We estimate the value of $\frac{LC}{m}$ from below.

Depending on the position of a point x and on hyperelliptic surface's type, we have the following possibilities for C to be an irreducible curve:

(1) $C \equiv B \equiv (0, k)$ and x is an arbitrary point, where $k = 1$ for a hyperelliptic surface of an odd type; $k = 2$ for a hyperelliptic surface of type 2 and 4; $k = 3$ for a hyperelliptic surface of type 6. Then

$$\frac{LC}{m} = \frac{k}{1} \geq 1.$$

(2) $C \equiv nA/\mu \equiv (n, 0)$ and a point x lies on a fibre nA/μ , where $n \in \{1, 2\}$ for a hyperelliptic surface of type 1 and 2; $n \in \{1, 2, 4\}$ for type 3 and 4; $n \in \{1, 3\}$ for type 5 and 6; $n \in \{1, 2, 3, 6\}$ for type 7. Then

$$\frac{LC}{m} = \frac{n}{1} \geq 1.$$

(3) $C \equiv (\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$, and x is an arbitrary point. Then by Bézout's Theorem, intersecting C with a fibre B and with an appropriate, depending on the position of the point x , fibre nA/μ , we get:

$$\frac{LC}{m} = \frac{\alpha + \beta}{m} \geq \begin{cases} 1, & \text{in case of a hyperelliptic surface of type 1, 3, 5, 7;} \\ \frac{1}{2} + \frac{1}{2}, & \text{in case of a hyperelliptic surface of type 2;} \\ \frac{1}{2} + \frac{1}{4}, & \text{in case of a hyperelliptic surface of type 4;} \\ \frac{1}{3} + \frac{1}{3}, & \text{in case of a hyperelliptic surface of type 6.} \end{cases}$$

Therefore $\frac{LC}{m} \geq 1$ for a hyperelliptic surface of type 1, 2, 3, 5 and 7.

Now let S be a surface of type 4 or 6. We consider two cases. If $m = 1$, then $\frac{LC}{m} = \frac{\alpha + \beta}{m} \geq \frac{2}{1} > 1$. If $m \geq 2$, then by genus formula $C^2 \geq m^2 - m$, by Hodge Index Theorem $(LC)^2 \geq L^2 C^2 = 2C^2 \geq 2(m^2 - m)$, and therefore $\frac{LC}{m} \geq \sqrt{\frac{2(m^2 - m)}{m^2}} = \sqrt{2 - \frac{2}{m}} \geq 1$.

Hence independently of the type of the hyperelliptic surface we have $\varepsilon(L, x) \geq 1$. Moreover, for every hyperelliptic surface's type, $\varepsilon(L, x) = 1$ for a point x on a fibre A/μ . Therefore $\varepsilon(L) = 1$. \square

By the proof of Theorem 3.1 we immediately obtain a corollary:

Corollary 3.2. *Let S be a hyperelliptic surface of an odd type. Let L be a line bundle of type $(1, 1)$ on S . Then the Seshadri constant of L at any $x \in S$ is computed by a fibre B , hence*

$$\varepsilon(L, x) = 1 \text{ for any } x \in S.$$

On the other hand, it is not true that on every hyperelliptic surface the equality $\varepsilon(L, x) = 1$ holds for every $x \in S$.

Proposition 3.3. *There exists a hyperelliptic surface S such that for a line bundle L of type $(1, 1)$*

$$\varepsilon(L, 1) > 1.$$

Proof. Let S be a hyperelliptic surface of type 2, and let L be a line bundle of type $(1, 1)$ on S . Let x be a very general point on S . We will prove that $\varepsilon(L, x) \geq \frac{4}{3}$.

Let $C \equiv (\alpha, \beta)$ be an irreducible curve passing through a given point $x \in S$ with multiplicity m , $m \geq 1$.

Let $m = 1$. Assume that $\frac{LC}{m} < \frac{4}{3}$. Then $LC < \frac{4}{3}$, hence $\alpha + \beta < \frac{4}{3}$ and thus, as α and β are nonnegative integers, $\alpha + \beta \leq 1$. Therefore either $(\alpha, \beta) \equiv (1, 0) \equiv A/2$, or $(\alpha, \beta) \equiv (0, 1) \equiv B/2$. A divisor $A/2$ does not pass through x , a divisor $B/2$ is not effective, a contradiction.

Now let $m \geq 2$. We have to prove that $\frac{LC}{m} \geq \frac{4}{3}$. Both sides are nonnegative, hence equivalently $(LC)^2 \geq \frac{16}{9}m^2$. By Hodge Index Theorem it is enough to show that $L^2C^2 \geq \frac{16}{9}m^2$. By Xu-type lemma (Lemma 2.8) we have that $C^2 \geq m^2 - m + 2$. Hence it is enough to prove that $2m^2 - 2m + 4 \geq \frac{16}{9}m^2$. Equivalently $(m - 3)(m - 6) \geq 0$. The inequality is satisfied for $m \neq 4, 5$. We consider these two cases separately.

Let $m = 4$. Suppose that $\frac{LC}{4} < \frac{4}{3}$. Hence $LC < \frac{16}{3}$, so $\alpha + \beta \leq 5$. On the other hand, by Xu-type lemma $2\alpha\beta = C^2 \geq m^2 - m + 2 = 14$, a contradiction.

For $m = 5$, if $\frac{LC}{5} < \frac{4}{3}$ then $\alpha + \beta \leq 6$. By Xu-type lemma $\alpha\beta \geq 11$, a contradiction. This completes the proof. \square

Using the same method as presented in Theorem 3.3 one can show that for a very general point x on a hyperelliptic surface of type 2 and for L of type $(1, 1)$, the Seshadri constant of L at x is greater than a constant slightly bigger than $\frac{4}{3}$. The proof splits in a large number of cases and therefore we decide not to present it here. However precise study of this example might support the idea that this Seshadri constant is irrational.

Now we will prove a lower bound for the global Seshadri constant of an arbitrary ample line bundle on hyperelliptic surface of type 1.

Theorem 3.4. *Let S be a hyperelliptic surface of type 1. Let L be an ample line bundle of type (a, b) on S . Then*

$$\varepsilon(L) = \min\{a, b\}.$$

Proof. Let S be a hyperelliptic surface of type 1, let $L \equiv (a, b)$. Let $C \equiv (\alpha, \beta)$ denote an irreducible curve passing through a given point x with multiplicity m , $m \geq 1$. Using

Bézout's Theorem we obtain:

$$\frac{LC}{m} = \frac{a\beta + b\alpha}{m} \geq \begin{cases} a, & \text{if } C \equiv B \text{ and } x \text{ is an arbitrary point;} \\ b, & \text{if } C \equiv A/2 \text{ and } x \text{ lies on the singular fibre } A/2; \\ 2b, & \text{if } C \equiv A \text{ and } x \text{ lies on the fibre } A; \\ a + b, & \text{if } C \equiv (\alpha, \beta) \text{ and } x \text{ lies on one of the singular fibres } A/2; \\ \frac{a}{2} + b, & \text{if } C \equiv (\alpha, \beta) \text{ and } x \text{ lies on one of the fibres } A. \end{cases}$$

Hence on a hyperelliptic surface of type 1

$$\varepsilon(L) = \min\{a, b\}. \quad \square$$

By the theorem above we see that on a hyperelliptic surface of type 1 the global Seshadri constant of an ample line bundle L is always submaximal, ie. smaller than $\sqrt{L^2}$.

Note that the method used in Theorem 3.4 does not work on hyperelliptic surfaces of other types. For hyperelliptic surfaces of type 1 the lower bound of $\frac{LC}{m}$, where a curve C is not a fibre, is always greater than the value of $\frac{LC}{m}$ for some fibre C . It is also easy to show for which fibre and for which point position the global Seshadri constant is actually reached. This is not the case for hyperelliptic surfaces of types 2-7.

For hyperelliptic surfaces of types 2-7, we have the following lower bound for the global Seshadri constant

Theorem 3.5. *Let S be a hyperelliptic surface of type greater than 1. Let L be an ample line bundle of type (a, b) on S . Then*

$$\varepsilon(L) \geq \min\{a, b\}.$$

Proof. We have that $L \equiv (a, b) \equiv \min\{a, b\} \cdot M + N$, where $M \equiv (1, 1)$ and N is nef. By definition of a Seshadri constant, for every $x \in S$

$$\varepsilon(L, x) \geq \min\{a, b\} \cdot \varepsilon(M, x) + \varepsilon(N, x) \geq \min\{a, b\} \cdot \varepsilon(M, x).$$

Hence by Theorem 3.1

$$\varepsilon(L) \geq \min\{a, b\} \cdot \varepsilon(M) = \min\{a, b\}. \quad \square$$

3.2. Multi-point Seshadri constants of ample line bundles on non-rational surfaces. In this section we present a lower bound for Seshadri constant at r points in very general position on hyperelliptic surfaces.

The lower bound for multi-point Seshadri constants obtained in Theorem 3.6 is not far from the upper bound. As mentioned before, it is well-known (see e.g. [PSC2009], Proposition 2.1.1) that for smooth projective surfaces

$$\varepsilon(L, r) \leq \sqrt{\frac{L^2}{r}}.$$

Biran-Nagata-Szemberg conjecture says that for any algebraic surface there exists $r_0 > 0$ such that for every $r > r_0$ in fact there is an equality $\varepsilon(L, r) = \sqrt{\frac{L^2}{r}}$.

Theorem 3.6. *Let S be a hyperelliptic surfaces. Let L be an ample line bundle on S . Then*

$$\varepsilon(L, r) \geq \sqrt{\frac{L^2}{r}} \sqrt{1 - \frac{1}{8r}}, \quad r \geq 2.$$

Proof. The claim follows immediately from Harbourne-Roé theorem (Theorem 2.3) with $\mu = 8$. The point is to check that the assumptions of this theorem are satisfied with that particular constant. Turning into details we need to check the following two conditions:

(1) for every integer $1 \leq m < 8$

$$\alpha_0(L, m^{[r]}) \geq m \sqrt{L^2 \left(r - \frac{1}{8} \right)};$$

(2) for every integer $1 \leq m < \frac{8}{r-1}$ and for every integer k with $k^2 < \frac{r}{r-1} \min\{m, m+k\}$

$$\alpha_0(L, m^{[r-1]}, m+k) \geq \frac{mr+k}{r} \sqrt{L^2 \left(r - \frac{1}{8} \right)}.$$

Ad. (1). For $m = 1, 2, \dots, 7$ we ask whether the inequality

$$\alpha_0(L, m^{[r]}) \geq m \sqrt{L^2 \left(r - \frac{1}{8} \right)}$$

is satisfied.

Let C be an irreducible curve computing $\alpha(L, m^{[r]})$. It suffices to show that

$$LC \geq m \sqrt{L^2 \left(r - \frac{1}{8} \right)}.$$

As L is ample, by Hodge Index Theorem it is enough to prove that

$$L^2 C^2 \geq m^2 L^2 \left(r - \frac{1}{8} \right).$$

We split the proof that $C^2 \geq m^2 \left(r - \frac{1}{8} \right)$ into two cases: $m = 1$ and $m > 1$.

For $m = 1$, we have $h^0(C) = \dim |C| + 1 \geq r + 1$. Moreover by Proposition 2.6 (3), $h^0(C) = \frac{C^2}{2}$. Hence $\frac{C^2}{2} \geq r + 1$. Therefore it is enough to show that

$$2r + 2 \geq r - \frac{1}{8}.$$

This condition is satisfied for every positive r .

Now let $2 \leq m \leq 7$. By Xu-type lemma (Lemma 2.9), $C^2 \geq rm^2 - m + 2$. Hence it is enough to show that

$$rm^2 - m + 2 \geq m^2 \left(r - \frac{1}{8} \right),$$

which is elementary.

Ad. (2). In the table below we write down all values of r , m and k satisfying conditions $1 \leq m < \frac{8}{r-1}$ and $k^2 < \frac{r}{r-1} \min\{m, m+k\}$.

r	$m < \frac{8}{r-1}$	possible k
2	1	1
	2	1, -1
	3	1, -1, 2
	4	1, -1, 2
	5	1, -1, 2, -2, 3
	6	1, -1, 2, -2, 3
	7	1, -1, 2, -2, 3
3	1	1
	2	1, -1
	3	1, -1, 2
4	1	1
	2	1, -1
5	1	1
6	1	1
7	1	1
8	1	1

We have omitted the case $k = 0$ in each row, as for $k = 0$ we get the inequality already proved in (1).

Using Hodge Index Theorem, analogously to (1) the condition to check is reduced to the following inequality

$$C^2 \geq \left(\frac{mr + k}{r} \right)^2 \left(r - \frac{1}{8} \right),$$

where C be an irreducible curve computing $\alpha_0(L, m^{[r-1]}, m+k)$.

Again we consider two cases: $m = 1$ and $m > 1$.

Let $m = 1$. Hence $k = 1$. Since Xu-type lemma (Lemma 2.9) implies that $C^2 \geq r + 3$, we easily obtain that

$$C^2 \geq \left(\frac{r+1}{r} \right)^2 \left(r - \frac{1}{8} \right).$$

Let $m > 1$. Hence $r \in \{2, 3, 4\}$. By Xu-type lemma $C^2 \geq (r-1)m^2 + (m+k)^2 - m + 2$, so it is enough to show that

$$(r-1)m^2 + (m+k)^2 - m + 2 \geq \left(\frac{mr+k}{r} \right)^2 \cdot \left(r - \frac{1}{8} \right)$$

holds. After reordering the terms we obtain the following inequality

$$8r^2k^2 - 8r^2m + 16r^2 + m^2r^2 + 2mrk - 8rk^2 + k^2 \geq 0.$$

Simple computations confirm that the last inequality is satisfied for all admissible $m > 1$, r and k . The proof is completed. \square

Remark 3.7. Note that the Theorem 3.6 holds also for abelian surfaces with $\rho = 1$.

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